Math 210C Lecture 14 Notes

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1 Long Exact Sequences of Homology and Chain Homotopy

This lecture was given by Jeremy Brightbill.

1.1 Long exact sequences of homology

If X is a topological space, we can associate to it the chain complex $C_(X) = \mathrm{span}_K\{\sigma : \Delta^n \to X\}$. We are trying to capture "homological information," information about whether X is built out of glued together simpler parts. We often get enough information for what we're interested in by looking at the homology $H_n(X)$.

Last time, we showed that homology is a functor: if we have a map $f_\cdot : A_\cdot \to B_\cdot$, this induces a homomorphism $(f_n)_* : H_m(A) \to H_n(B)$.

Theorem 1.1. *Let*

$$0 \longrightarrow X_\cdot \xrightarrow{f} Y_\cdot \xrightarrow{g} Z_\cdot \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there exist maps $\delta_i : H_i(Z_\cdot) \to H_{i-1}(X_\cdot)$ such that the sequence

$$\cdots \longrightarrow H_i(X) \xrightarrow{f_*} H_i(Y) \xrightarrow{g_*} H_i(Z) \xrightarrow{f_i} H_{i-1}(X_\cdot) H_{i-1}(Y) \longrightarrow \cdots$$

is exact.

Proof. We have the diagram

$$0 \longrightarrow X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \longrightarrow 0$$

$$0 \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} Y_{i-1} \xrightarrow{g_{i-1}} Z_{i-1} \longrightarrow 0$$

Using the snake lemma, we get exact sequences

$$0 \longrightarrow \ker(d_{X_i}) \longrightarrow \ker(d_{Y_i}) \longrightarrow \ker(d_{Z_i})$$

$$\operatorname{coker}(d_{X_i}) \longrightarrow \operatorname{coker}(d_{Y_i}) \longrightarrow \operatorname{coker}(d_{Z_i}) \longrightarrow 0$$

So we have

$$\begin{array}{ccccccc}
 & & \operatorname{coker}(d_{X_{i+2}}) & \longrightarrow & \operatorname{coker}(d_{Y_{i+2}}) & \longrightarrow & \operatorname{coker}(d_{Z_{i+2}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{d}_{X_{i+1}}} & & \downarrow{\scriptstyle \overline{d}_{Y_{i+1}}} & & \downarrow{\scriptstyle \overline{d}_{Z_{i+1}}} & & \\
0 & \longrightarrow & \ker(d_{X_i}) & \longrightarrow & \ker(d_{Y_i}) & \longrightarrow & \ker(d_{Z_i}) & &
\end{array}$$

Using the snake lemma again, we get an exact sequence

$$\ker(\overline{d}_{X_{i+1}}) \longrightarrow \ker(\overline{d}_{Y_{i+1}}) \longrightarrow \ker(\overline{d}_{Z_{i+1}}) \longrightarrow \operatorname{coker}(\overline{d}_{X_{i+1}}) \longrightarrow \operatorname{coker}(\overline{d}_{Y_{i+1}}) \longrightarrow \operatorname{coker}(\overline{d}_{Z_{i+1}})$$

Now observe that $\ker(\overline{d}_{X_{i+1}}) = H_{i+1}(X)$, $\ker(\overline{d}_{Y_{i+1}}) = H_{i+1}(Y)$, and $\ker(\overline{d}_{Z_{i+1}}) = H_{i+1}(Z)$, $\operatorname{coker}(\overline{d}_{X_{i+1}}) = H_i(X)$, $\operatorname{coker}(\overline{d}_{Y_{i+1}}) = H_i(Y)$, and $\operatorname{coker}(\overline{d}_{Z_{i+1}}) = H_i(Z)$. □

Example 1.1. Let $X_.$ be a complex. Define

$$\tau_{\geq 0}X_. = \quad \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow \ker(d_{X_0}) \longrightarrow 0$$

We get the morphism of complexes

$$\begin{array}{ccccccccccccc}
\tau_{\geq 0}X_. : & & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & \ker(d_{X_0}) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
X_. : & & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
\tau_{<0}X_. : & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{im}(d_{X_0}) & \longrightarrow & X_{-1} & \longrightarrow & \cdots
\end{array}$$

where $\tau_{<0}X_. = X_./\tau_{\geq 0}X_.$. Then $H_i(\tau_{<0}X) = H_i(X)$ for all $i \geq 0$, and $H_i(\tau_{\geq 0}X) = 0$ for all $i < 0$. We have

$$\begin{array}{l}
0 \longrightarrow H_1(\tau_{\geq 0}X_.) \longrightarrow H_1(X) \longrightarrow \underbrace{H_1(\tau_{<0}X_.)}_{=0} \\
\longrightarrow H_0(\tau_{\geq 0}X_.) \longrightarrow H_0(X) \longrightarrow \underbrace{H_0(\tau_{<0}X_.)}_{=0} \\
\longrightarrow \underbrace{H_{-1}(\tau_{\geq 0}X_.)}_{=0} \longrightarrow H_{-1}(X) \longrightarrow H_{-1}(\tau_{<0}X_.) \longrightarrow 0
\end{array}$$

Example 1.2. Call

$$X^{\geq 0} = \quad \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow 0$$

Then we get a morphism of complexes:

$$\begin{array}{ccccccccccc}
X^{<0}: & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X_{-1} & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
X_{.}: & & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
X^{\geq 0}: & & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

We get

$$\begin{array}{ccccc}
\underbrace{H_1(X^{<0})}_{=0} & \longrightarrow & H_1(X) & \longrightarrow & H_1(X^{\geq 0}) \\
 & & & \swarrow & \\
\underbrace{H_0(X^{<0})}_{=0} & \longrightarrow & H_0(X) & \longrightarrow & H_0(X^{\geq 0}) \\
 & & & \swarrow & \\
H_{-1}(X^{<0}) & \longrightarrow & H_{-1}(X) & \longrightarrow & \underbrace{H_{-1}(X^{\geq 0})}_{=0}
\end{array}$$

1.2 Chain homotopy

Nobody works in the category of chain complexes for very long. It's too hard for chain complexes to be isomorphic.

Definition 1.1. Let $f_., g_. : X \to Y$ be morphisms of complexes. Write $f \sim g$ if there exist $h_i : X_i \to Y_i$ such that $f - g = d_{Y_{i+1}} h_i + h_{i-1} d_{X_i}$.

$$\begin{array}{ccccc}
X_{i+1} & \longrightarrow & X_i & \longrightarrow & X_{i-1} \\
{\scriptstyle f_{i+1}}\downarrow{\scriptstyle g_{i+1}} & \overset{h_i}{\swarrow} & {\scriptstyle f_i}\downarrow{\scriptstyle g_i} & \overset{h_{i-1}}{\swarrow} & {\scriptstyle f_{i-1}}\downarrow{\scriptstyle g_{i-1}} \\
Y_{i+1} & \longrightarrow & Y_i & \longrightarrow & Y_{i-1}
\end{array}$$

We say that f is **chain homotopic** to g

If X and Y are topological spaces with $f, g : X \to Y$ homotopic, then $f_* \sim g_*$ are homotopic, where $f_*, g_* : C_.(X) \to C_.(Y)$.

Proposition 1.1. *Chain homotopy has the following properties:*

1. *Chain homotopy is an equivalence relation on* $\mathrm{Hom}(X_\cdot, Y_\cdot)$.

2. *Let* $f, g : X_\cdot \to Y_\cdot$, $a_\cdot : W_\cdot \to X_\cdot$, *and* $b_\cdot : Y_\cdot \to Z_\cdot$. *If* $f \sim g$, *then* $f \circ a = g \circ a$ *and* $b \circ f \sim b \circ g$.

3. *If* $f \sim g$ *and* $f' \sim g'$, *then* $f + g \sim f' + g'$, *and* $-f \sim -g$.

Definition 1.2. The **homotopy category of complexes** $\mathrm{Ho}(\mathscr{A})$ has objects te chain complexes and $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{A})}(X_\cdot, Y_\cdot) = \mathrm{Hom}_{\mathrm{Ch}(A)}(X_\cdot, Y_\cdot)/\{f \sim g\}$.

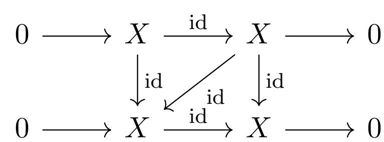
Definition 1.3. We say that f is **null-homotopic** if $f \sim 0$.

Definition 1.4. $f : X \to Y$ and $g : Y \to X$ are **homotopy equivalences** if $1_X \sim gf$ and $1_Y \sim fg$.

Proposition 1.2. *Let* $f \sim g : X \to Y$. *Then* $f_{i*} = g_{i*} : X_i(X) \to H_i(Y)$.

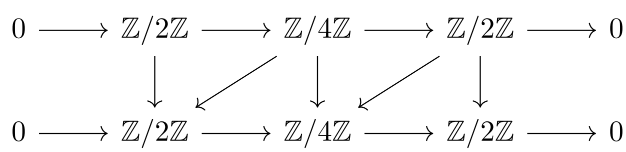
Proof. $(f - g)_* = f_* - g_*$, so it suffices to prove this for $f \sim 0$. Then $d = dh + hd = 0$ on $H_i(X)$. □

Example 1.3. In the diagram



$\mathrm{id}_{X_\cdot} \sim 0$. So $X_\cdot \cong 0$ in the homotopy category.

Example 1.4. In the diagram



id_X is not homotopic to 0.